

UPPER AND LOWER DENSITIES HAVE THE STRONG DARBOUX PROPERTY

PAOLO LEONETTI AND SALVATORE TRINGALI

ABSTRACT. Let $\mathcal{P}(\mathbf{N})$ be the power set of \mathbf{N} . An upper density (on \mathbf{N}) is a nondecreasing and subadditive function $\mu^* : \mathcal{P}(\mathbf{N}) \rightarrow \mathbf{R}$ such that $\mu^*(\mathbf{N}) = 1$ and $\mu^*(k \cdot X + h) = \frac{1}{k}\mu^*(X)$ for all $X \subseteq \mathbf{N}$ and $h, k \in \mathbf{N}^+$, where $k \cdot X + h := \{kx + h : x \in X\}$.

The upper asymptotic, upper Banach, upper logarithmic, upper Buck, upper Pólya, and upper analytic densities, together with the upper α -densities, are some remarkable examples of upper densities in the sense of the above definition.

We show that every upper density μ^* has the strong Darboux property, and so does the associated lower density $\mu_*(\mathbf{N}) : \mathcal{P}(\mathbf{N}) \rightarrow \mathbf{R} : X \mapsto 1 - \mu^*(\mathbf{N} \setminus X)$, where a function $f : \mathcal{P}(\mathbf{N}) \rightarrow \mathbf{R}$ is said to have the strong Darboux property if, whenever $X \subseteq Y \subseteq \mathbf{N}$ and $a \in [f(X), f(Y)]$, there exists a set A such that $X \subseteq A \subseteq Y$ and $f(A) = a$.

In fact, we prove the above under the assumption that the monotonicity of μ^* is relaxed to the weaker condition that $\mu^*(X) \leq 1$ for every $X \subseteq \mathbf{N}$. In addition, we work out several examples to answer some related questions and argue in favor of the “sharpness” of our results.

1. INTRODUCTION

Motivated by unpublished work of S. Révész and I. Ruzsa on a “general theory of densities” for groups, the authors have recently introduced, and studied some fundamental aspects of, an “axiomatic theory of densities” [11], tailored to the integers and built around the properties of certain (set) functions called “upper [quasi-]densities”. These functions are also the subject of the present manuscript, as our main goal is to prove that they satisfy a kind of “intermediate value property” we refer to as the strong Darboux property (see Section 3 for details).

Throughout, we will let \mathbf{H} be either \mathbf{Z} , \mathbf{N} , or \mathbf{N}^+ , and μ^* a function $\mathcal{P}(\mathbf{H}) \rightarrow \mathbf{R}$; we regard \mathbf{H} as a sort of “parameter” allowing for different scenarios and some flexibility, though it makes almost no difference for this paper to focus on the case $\mathbf{H} = \mathbf{N}$ (see Section 2 for notation and terminology used, but not defined, in this introduction).

We start by recalling some basic facts and definitions from [11]. In particular, we say that μ^* is an *upper density* (on \mathbf{H}) if, for all $X, Y \subseteq \mathbf{H}$ and $h, k \in \mathbf{N}^+$, we have:

- (F1) $\mu^*(\mathbf{H}) = 1$;
- (F2) $\mu^*(X) \leq \mu^*(Y)$ for $X \subseteq Y$;
- (F3) $\mu^*(X \cup Y) \leq \mu^*(X) + \mu^*(Y)$;

2010 *Mathematics Subject Classification.* Primary 11B05, 28A10. Secondary 39B52, 60B99.

Key words and phrases. Asymptotic (or natural) density, Banach (or uniform) density, Darboux (or intermediate value) property, set functions, upper and lower densities (and quasi-densities).

$$(F4) \quad \mu^*(k \cdot X) = \frac{1}{k} \mu^*(X), \text{ where } k \cdot X := \{kx : x \in X\};$$

$$(F5) \quad \mu^*(X + h) = \mu^*(X).$$

For future reference, notice that axioms (F4) and (F5) together are equivalent to the following:

$$(F4^b) \quad \mu^*(k \cdot X + h) = \frac{1}{k} \mu^*(X) \text{ for all } X \subseteq \mathbf{H} \text{ and } h, k \in \mathbf{N}^+.$$

To ease the exposition, we will occasionally say that μ^* is: monotone if it satisfies (F2); subadditive if it satisfies (F3); additive if $\mu^*(X \cup Y) = \mu^*(X) + \mu^*(Y)$ whenever $X, Y \subseteq \mathbf{H}$ and $X \cap Y$ is empty; (-1) -homogeneous if it satisfies (F4); and shift-invariant (translational invariant, or translational symmetric) if it satisfies (F5).

On the other hand, we call μ^* an *upper quasi-density* (on \mathbf{H}) if it satisfies (F1), (F3), (F4^b), and the next condition, which is clearly implied by (F1) and (F2):

$$(F2^b) \quad \mu^*(X) \leq 1 \text{ for every } X \subseteq \mathbf{H}.$$

Of course, every upper density is an upper quasi-density, and [11, Theorem 1] proves that non-monotone upper quasi-densities do actually exist, while it is shown in [11, Sections 3 and 4 and Examples 4-7] that each of the following is an upper density in the sense of the above definitions:

- the upper asymptotic (or natural) density (on \mathbf{H}), namely the function

$$\mathcal{P}(\mathbf{H}) \rightarrow \mathbf{R} : X \mapsto \limsup_{n \rightarrow \infty} \frac{|X \cap [1, n]|}{n};$$

- the upper α -density, with α a fixed real parameter ≥ -1 , i.e. the function

$$\mathcal{P}(\mathbf{H}) \rightarrow \mathbf{R} : X \mapsto \limsup_{n \rightarrow \infty} \frac{\sum_{i \in X \cap [1, n]} i^\alpha}{\sum_{i \in [1, n]} i^\alpha};$$

- the upper Banach (or uniform) density, that is the function

$$\mathcal{P}(\mathbf{H}) \rightarrow \mathbf{R} : X \mapsto \lim_{n \rightarrow \infty} \max_{h \geq 0} \frac{|X \cap [h+1, h+n]|}{n};$$

- the upper analytic density, to wit the function

$$\mathcal{P}(\mathbf{H}) \rightarrow \mathbf{R} : X \mapsto \limsup_{s \rightarrow 1^+} \frac{1}{\zeta(s)} \sum_{i \in X^+} \frac{1}{i^s},$$

where ζ is the restriction to the interval $]1, \infty[$ of the Riemann zeta function;

- the upper Buck density, viz. the function

$$\mathcal{P}(\mathbf{H}) \rightarrow \mathbf{R} : X \mapsto \inf_{S \in \mathcal{A} : X \subseteq S} \mathbf{d}^*(S),$$

where \mathbf{d}^* is the upper asymptotic density (see above) and \mathcal{A} the set of all finite unions of arithmetic progressions of \mathbf{H} ;

- the upper Pólya density, i.e. the function

$$\mathcal{P}(\mathbf{H}) \rightarrow \mathbf{R} : X \mapsto \lim_{s \rightarrow 1^-} \limsup_{n \rightarrow \infty} \frac{|X \cap [1, n]| - |X \cap [1, ns]|}{(1-s)n}.$$

All of these densities, along with many others, have been the subject of a great deal of research, and some of them have played a prominent role in the development of (probabilistic and additive) number theory and some areas of analysis and ergodic theory.

Roughly speaking, one reason for this is that densities provide an effective alternative to measures when it comes to the study of the interrelation between the “structure” of a set of integers and some kind of information about its “largeness”, a general principle that is brightly illustrated by the Erdős-Turán conjecture [19, Section 35.4] that any set X of positive integers such that $\sum_{x \in X} \frac{1}{x} = \infty$ contains arbitrarily long (finite) arithmetic progressions; see also [11] and references therein for some other pointers to the literature.

Consider now the function $\mu_* : \mathcal{P}(\mathbf{H}) \rightarrow \mathbf{R} : X \mapsto 1 - \mu^*(X^c)$. We call μ_* the conjugate of μ^* , and actually the lower dual of μ^* if $\mu_*(X) \leq \mu^*(X)$ for every $X \subseteq \mathbf{H}$, which is especially the case when μ^* is subadditive, as noted in [11, Proposition 2(vi)]. Accordingly, we refer to μ_* as a lower [quasi-]density (on \mathbf{H}), or more specifically as the lower [quasi-]density associated to μ^* , provided that μ^* is an upper [quasi-]density. E.g., the lower dual of the upper asymptotic density (on \mathbf{H}) is the lower asymptotic density, viz. the function

$$\mathcal{P}(\mathbf{H}) \rightarrow \mathbf{R} : X \mapsto \liminf_{n \rightarrow \infty} \frac{|X \cap [1, n]|}{n},$$

and the lower dual of the upper Buck density is the lower Buck density, that is the function

$$\mathcal{P}(\mathbf{H}) \rightarrow \mathbf{R} : X \mapsto \sup_{S \in \mathcal{A} : S \subseteq X} d^*(S),$$

where d^* is, again, the upper asymptotic density.

It follows from [11, Theorem 2] that if μ^* is an upper quasi-density then the image set of μ^* , and hence also of μ_* , is the entire interval $[0, 1]$, and the ultimate goal of the present paper is to prove that these results can be considerably strengthened (see Theorem 3).

Before proceeding to details, we feel, however, that a minor remark is in order: While it is arguable that non-monotone quasi-densities are not very interesting from the point of view of applications (whatever this may mean), it seems meaningful to establish if certain properties of a specific class of objects depend or not on a particular assumption (in the present case of interest, the axiom of monotonicity), as this provides, we believe, a better understanding of the objects at hand. That is basically our motivation for considering upper quasi-densities, instead of restricting our attention to upper densities.

2. NOTATION AND CONVENTIONS

We refer to [11], and particularly to [11, Sections 2 and 3], for most notation, terminology and conventions used through this work. Notice that measures herein will always be real-valued, but unless a statement to the contrary is made, they do not need to be nonnegative or countably additive (that is, they may be signed or finitely additive).

We write \mathbf{R} , \mathbf{Z} , and \mathbf{N} , respectively, for the sets of reals, integers, and nonnegative integers, and unless otherwise specified, we use the letters h , i , and k , with or without subscripts, for nonnegative integers, the letter n for a positive integer, and the letter s for a positive real.

Given $a, b \in \mathbf{R} \cup \{\infty\}$, we let $[a, b] := \{x \in \mathbf{R} \cup \{\infty\} : a \leq x \leq b\}$ and $\llbracket a, b \rrbracket := [a, b] \cap \mathbf{Z}$, and we denote by $]a, b]$, $[a, b[$ and $]a, b[$, respectively, the intervals $[a, b] \setminus \{a\}$, $[a, b] \setminus \{b\}$ and $[a, b] \setminus \{a, b\}$ (we use ∞ in place of $+\infty$). Moreover, for $X \subseteq \mathbf{R}$ we set $X^+ := X \cap]0, \infty[$, so in particular $\mathbf{R}^+ =]0, \infty[$ and $\mathbf{N}^+ = \mathbf{N} \setminus \{0\}$.

For a set S we denote by $\mathcal{P}(S)$ the power set of S and by $|S|$ its size, i.e. $|S| = k$ if S is finite and has k elements, while $|S| = \infty$ otherwise. If $f : X \rightarrow Y$ is a function, we use $\text{Im}(f)$ for the image set of f , namely $\text{Im}(f) := \{f(x) : x \in X\} \subseteq Y$.

We take an arithmetic progression of \mathbf{H} to be a set of the form $k \cdot \mathbf{H} + h$ with $k \in \mathbf{H} \setminus \{0\}$ and $h \in \mathbf{H} \cup \{0\}$. We assume the convention that an empty union is equal to \emptyset and an empty sum of real numbers is 0. Moreover, we say that a sequence $(x_n)_{n \geq 1}$ is the natural enumeration of a set $X \subseteq \mathbf{N}$ if $X = \{x_n : n \in \mathbf{N}^+\}$ and $x_n < x_{n+1}$ for each $n \in \mathbf{N}^+$.

Lastly, we let $\lceil \cdot \rceil$ denote the ceil function, i.e. the function $\mathbf{R} \rightarrow \mathbf{Z} : x \mapsto \min\{k \in \mathbf{Z} : x \leq k\}$, and for a set $X \subseteq \mathbf{H}$ we take $X^c := \mathbf{H} \setminus X$.

3. DARBOUX PROPERTIES

We start this section with a couple more of definitions. Given a set S , we say that a partial function $f : \mathcal{P}(S) \rightarrow \mathbf{R}$ with domain \mathcal{D} has:

- (D1) the *weak Darboux property* if $\emptyset \in \mathcal{D}$ and for every $X \in \mathcal{D}$ and $a \in [f(\emptyset), f(X)]$ there is a set $A \in \mathcal{D}$ such that $A \subseteq X$ and $f(A) = a$;
- (D2) the *strong Darboux property* if for all $X, Y \in \mathcal{D}$ with $X \subseteq Y$ and every $a \in [f(X), f(Y)]$ there exists a set $A \in \mathcal{D}$ such that $X \subseteq A \subseteq Y$ and $f(A) = a$.

Of course, (D1) is implied by (D2) provided that $\emptyset \in \mathcal{D}$, and the converse is true, e.g., of finitely additive measures. Notice also that, since f does not need to be monotone in condition (D1), it may well happen that $f(X) < f(\emptyset)$ for some $X \in \mathcal{D}$, in which case $[f(\emptyset), f(X)]$ is empty and there is nothing to prove; analogous considerations apply to the strong Darboux property.

Some authors, either in measure theory, see, e.g., [10, Chapter V, Section 46.I, Corollary 3'] and [3, Chapter I, Section 2.9, Definition 4], or in connection to the study of densities in number theory, see, for instance, [16, Section 2], [12, p. 217], and [9], refer to (D1) as simply the Darboux property (notice that [16] points to [3], [12] points to [10], and [9] points to [12] as a source for the terminology). However, that does not sound very fit to us, as (D2) is arguably closer than (D1) to the spirit of the intermediate value property of real-valued functions of a real variable, so we prefer sticking to our own definitions.

Other terms of common usage to allude to condition (D1), particularly in the literature on charges, are “strongly non-atomic”, see, e.g., [2, Definition 5.1.5], and “full-valued” (or “full valued”), see , where the focus is actually on finite nonnegative charges.

We note that if f has the weak Darboux property then $f(X) \leq f(\emptyset)$ for every finite $X \subseteq S$: If $f(\emptyset) < f(X)$ for some $X \subseteq S$, then the interval $[f(\emptyset), f(X)]$ has positive width, so X should have infinitely many subsets for f to have the weak Darboux property, which, however, is not the case when X is finite. In addition, we have the following elementary result:

Proposition 1. *Let μ^* be a function $\mathcal{P}(\mathbf{H}) \rightarrow \mathbf{R}$ and μ_* its conjugate. Then μ^* has the strong Darboux (respectively, weak Darboux) property if and only if μ_* does.*

Proof. Since μ^* is the conjugate of μ_* , it is enough to assume, as we do, that μ^* has the strong Darboux (respectively, weak Darboux) property and to prove that the same is true of μ_* .

For this, fix $X, Y \subseteq \mathbf{H}$ with $X \subseteq Y$ and $a \in [\mu_*(X), \mu_*(Y)]$, and set $X := \emptyset$ if μ^* has just the weak Darboux property. Then $Y^c \subseteq X^c$, and it follows from the hypotheses, and from the fact that $\mu_*(S) + \mu^*(S^c) = 1$ for every $S \subseteq \mathbf{H}$, that there exists $A \subseteq \mathbf{H}$ such that $Y^c \subseteq A \subseteq X^c$ and $1 - \mu_*(A^c) = \mu^*(A) = 1 - a$. Hence $\mu_*(A^c) = a$, which ultimately yields that μ_* has the strong Darboux (respectively, weak Darboux) property, when considering that $X \subseteq A^c \subseteq Y$. ■

With this in mind, here comes the main contribution of the present paper, which is reminiscent of results of D. Maharam [14, Theorem 2], on finitely additive nonnegative measures on \mathbf{N}^+ , and G. Sikorski [18, Problem 12, p. 225], on non-atomic, countably additive, nonnegative measures¹.

Main Theorem. *Every upper quasi-density has the strong Darboux property.*

The theorem, which is proved in Section 4, provides a partial answer to [11, Question 5] and leads, together with Proposition 1, to the following results (we omit further details):

Corollary 1. *Every lower quasi-density has the strong Darboux property.*

In particular, the next theorem is now obvious, but we record it here for future reference:

Corollary 2. *Upper and lower quasi-densities have the weak Darboux property.*

Special instances of Corollary 2 have already appeared in the literature, the proofs of these former results being based on ad hoc arguments tailored to the particular densities under consideration (contrarily to the proof of Theorem 3).

To be more precise, it is known from work of G. Grekos, see [6, 7], that the upper α -densities (on \mathbf{N}^+) have the weak Darboux property for every real $\alpha \geq -1$, and this has been later extended (and in a stronger form) to certain weighted densities (on \mathbf{N}^+), see [9, Proposition 1].

4. PROOF OF MAIN THEOREM

We will need the following result from [11], which establishes that upper quasi-densities, though not necessarily monotone, satisfy a kind of “weak monotonicity”:

Proposition 2. *Let μ^* be an upper quasi-density on \mathbf{H} , and pick $X, Y \in \mathcal{P}(\mathbf{H})$ such that $X \subseteq Y$ and Y is a finite union of arithmetic progressions of \mathbf{H} , or differs from a set of this form by finitely many integers. Then $\mu^*(X) \leq \mu^*(Y)$.*

¹: Incidentally, the latter result is frequently attributed to W. Sierpiński, and in so doing a reference to [17] is often provided as a “proof”. Yet, this seems to be wrong, as Sierpiński’s paper deals with the range of certain finitely additive measures $\mathcal{P}(S) \rightarrow \mathbf{R}$ for which S is a bounded subset of \mathbf{R}^n . In fact, the mistake may have originated from an incorrect interpretation of a footnote on the very same page of Sikorski’s book [18], where the reader is addressed to [17] and [4], but most likely for the sake of comparison.

Proof. See [11, Proposition 6] for details. ■

We need Proposition 2 in the lemma below, which in turn is used in the proof of Theorem 3 and may be of independent interest, in that it can be probably adapted to prove other results along the lines of Theorem 3, but for different classes of “densities” than the ones picked up in this work.

For ease of notation, we will denote by $\mathcal{V}_{k,\mathcal{H}}$, for all $k \in \mathbf{N}^+$ and $\mathcal{H} \subseteq \mathbf{N}$, the set $\bigcup_{h \in \mathcal{H}}(k \cdot \mathbf{H} + h)$, and write \mathcal{V}_k in place of $\mathcal{V}_{k,\mathcal{H}}$ when $\mathcal{H} = \llbracket 0, k-1 \rrbracket$.

Lemma 1. *Let μ^* be an upper quasi-density on \mathbf{H} , pick $A, B \subseteq \mathbf{H}$ with $A \subseteq B$ and $\mu^*(A) < \mu^*(B)$, and fix $a, b \in \mathbf{R}$ such that $\mu^*(A) \leq a < b \leq \mu^*(B)$. Then, for every $k > (b-a)^{-1}$ there exist $\mathcal{H}_0 \subseteq \llbracket 0, k-1 \rrbracket$ and $h_0 \in \llbracket 0, k-1 \rrbracket$ with the property that*

$$a < \mu^*(A \cup (B \cap \mathcal{V}_{k,\mathcal{H}_0})) < b \leq \mu^*(A \cup (B \cap \mathcal{V}_{k,\mathcal{H}_0 \cup \{h_0\}})).$$

Proof. Fix an integer $k > (b-a)^{-1}$ and let \mathcal{L}_k be the set of all subsets \mathcal{H} of $\llbracket 0, k-1 \rrbracket$ for which $\mu^*(A \cup (B \cap \mathcal{V}_{k,\mathcal{H}})) > a$. We have that $\mathbf{H} = S \cup \mathcal{V}_k$ for some finite set S ; in fact, $S = \llbracket 1, k-1 \rrbracket$ if $\mathbf{H} = \mathbf{N}^+$, and $S = \emptyset$ otherwise. So it follows from (F3) and Proposition 2 that

$$\begin{aligned} a < \mu^*(B) &= \mu^*(A \cup B) = \mu^*(A \cup (B \cap (S \cup \mathcal{V}_k))) = \mu^*(A \cup (B \cap \mathcal{V}_k) \cup (B \cap S)) \\ &\leq \mu^*(A \cup (B \cap \mathcal{V}_k)) + \mu^*(B \cap S) = \mu^*(A \cup (B \cap \mathcal{V}_k)). \end{aligned} \tag{1}$$

This implies that $\llbracket 0, k-1 \rrbracket \in \mathcal{L}_k$. Let \mathcal{H}_a be a set of minimal cardinality in \mathcal{L}_k (which exists since \mathcal{L}_k is finite and nonempty), and observe that \mathcal{H}_a is nonempty, otherwise we would have that $\mathcal{V}_{k,\mathcal{H}_a} = \emptyset$, and hence $a < \mu^*(A \cup (B \cap \mathcal{V}_{k,\mathcal{H}_a})) = \mu^*(A) \leq a$, which is absurd.

Now, suppose for the sake of a contradiction that $\mu^*(A \cup (B \cap \mathcal{V}_{k,\mathcal{H}_a})) \geq b$, and using that $\mathcal{H}_a \neq \emptyset$, fix $h \in \mathcal{H}_a$. Since $|\mathcal{H}_a \setminus \{h\}| < |\mathcal{H}_a|$ and \mathcal{H}_a is, by construction, an element of minimal cardinality in \mathcal{L}_k , we must have that $\mu^*(A \cup (B \cap \mathcal{V}_{k,\mathcal{H}_a \setminus \{h\}})) \leq a$. But it holds that

$$A \cup (B \cap \mathcal{V}_{k,\mathcal{H}_a}) = A \cup (B \cap \mathcal{V}_{k,\mathcal{H}_a \setminus \{h\}}) \cup (B \cap (k \cdot \mathbf{H} + h)),$$

and so we get from (F3), (F4^b) and [11, Proposition 9] that

$$\begin{aligned} b &\leq \mu^*(A \cup (B \cap \mathcal{V}_{k,\mathcal{H}_a})) \leq \mu^*(A \cup (B \cap \mathcal{V}_{k,\mathcal{H}_a \setminus \{h\}})) + \mu^*(B \cap (k \cdot \mathbf{H} + h)) \\ &\leq a + \mu^*(k \cdot \mathbf{H} + h) = a + \frac{1}{k} < a + b - a = b, \end{aligned}$$

which is a contradiction. It follows that $a < \mu^*(A \cup (B \cap \mathcal{V}_{k,\mathcal{H}_a})) < b$.

At this point, denote by \mathcal{M}_k the set of all subsets \mathcal{H} of $\llbracket 0, k-1 \rrbracket$ containing \mathcal{H}_a and such that $a < \mu^*(A \cup (B \cap \mathcal{V}_{k,\mathcal{H}})) < b$; we know from the above that $\mathcal{H}_a \in \mathcal{M}_k$. Then, let \mathcal{H}_0 be a set of maximal cardinality in \mathcal{M}_k (which exists since \mathcal{M}_k , too, is finite and nonempty), and \mathcal{S}_k the set of all subsets \mathcal{H} of $\llbracket 0, k-1 \rrbracket$ containing \mathcal{H}_0 and such that $\mu^*(A \cup (B \cap \mathcal{V}_{k,\mathcal{H}})) \geq b$; we have from (1) that $\llbracket 0, k-1 \rrbracket \in \mathcal{S}_k$, as $\mu^*(A \cup (B \cap \mathcal{V}_k)) \geq \mu^*(B) \geq b$. Accordingly, let \mathcal{H}_b be an element of minimal cardinality in \mathcal{S}_k (which exists, once again, since \mathcal{S}_k is finite and nonempty).

It is clear that $|\mathcal{H}_b| \geq 1 + |\mathcal{H}_0|$, because $\mathcal{H}_0 \subseteq \mathcal{H}_b$ and $\mu^*(\mathcal{H}_0) < b \leq \mu^*(\mathcal{H}_b)$, and consequently $\mathcal{H}_0 \subsetneq \mathcal{H}_b$; we claim that $|\mathcal{H}_b| = |\mathcal{H}_0| + 1$. In fact, suppose to the contrary that $|\mathcal{H}_b| \geq 2 + |\mathcal{H}_0|$,

and pick $h \in \mathcal{H}_b \setminus \mathcal{H}_0$. Then $\mathcal{H}_0 \subsetneq \mathcal{H}_b \setminus \{h\} \subsetneq \mathcal{H}_b$, so \mathcal{H}_0 being an element of maximal cardinality in \mathcal{M}_k and \mathcal{H}_b an element of minimal cardinality in \mathcal{S}_k entail that $\mathcal{H}_b \setminus \{h\} \notin \mathcal{M}_k \cup \mathcal{S}_k$.

Thus $\mu^*(A \cup (B \cap \mathcal{V}_{k, \mathcal{H}_b \setminus \{h\}})) \leq a$, which is however impossible by the same argument used above to prove that $\mu^*(A \cup (B \cap \mathcal{V}_{k, \mathcal{H}_a})) < b$, as it would imply (we omit some details) that

$$b \leq \mu^*(A \cup (B \cap \mathcal{V}_{k, \mathcal{H}_b})) \leq \mu^*(A \cup (B \cap \mathcal{V}_{k, \mathcal{H}_b \setminus \{h\}})) + \mu^*(B \cap (k \cdot \mathbf{H} + h)) < b.$$

This completes the proof of the lemma, as it shows that, letting h_0 be the unique element in $\mathcal{H}_b \setminus \mathcal{H}_0$, we have $a < \mu^*(A \cup (B \cap \mathcal{V}_{k, \mathcal{H}_0})) < b \leq \mu^*(A \cup (B \cap \mathcal{V}_{k, \mathcal{H}_0 \cup \{h_0\}}))$. \blacksquare

We are now ready to prove the main result of the paper.

Proof of Main Theorem. Let μ^* be an upper quasi-density on \mathbf{H} , and fix $X, Y \in \mathcal{P}(\mathbf{H})$, $X \subseteq Y$. If $\mu^*(Y) \leq \mu^*(X)$, the conclusion is trivial, so assume in what follows that $\mu^*(X) < \mu^*(Y)$ and let $a \in]\mu^*(X), \mu^*(Y)[$ (the boundary cases are obvious).

CLAIM. There exist two sequences $(A_n)_{n \geq 1}$ and $(B_n)_{n \geq 1}$ of subsets of \mathbf{H} for which:

- (i) $X \subseteq A_1 \subseteq \dots \subseteq A_n \subseteq \dots \subseteq B_n \subseteq \dots \subseteq B_1 \subseteq Y$;
- (ii) $\mu^*(A_n) < a \leq \mu^*(B_n)$ for all $n \in \mathbf{N}^+$;
- (iii) Given $n \in \mathbf{N}^+$, there exist $h, k \in \mathbf{N}$ such that $k \geq n$ and $B_n \setminus A_n \subseteq k \cdot \mathbf{H} + h$.

Proof of the claim. To begin, set $A_1 := X$ and $B_1 := Y$, by noting that $\mu^*(X) < a \leq \mu^*(Y) \leq 1 \leq 1 + \mu^*(X)$ and $Y \setminus X \subseteq 1 \cdot \mathbf{H} + 0$. Next, fix $v \in \mathbf{N}^+$ and suppose for the sake of induction that we have already found subsets A_1, \dots, A_v and B_1, \dots, B_v of \mathbf{H} such that $A_1 \subseteq \dots \subseteq A_v \subseteq B_v \subseteq \dots \subseteq B_1$ and conditions (ii) and (iii) hold true for $1 \leq n \leq v$.

Let $k > (a - \mu^*(A_v))^{-1}$. By our assumptions, we have that $\mu^*(A_v) < \mu^*(B_v)$ and $\mu^*(B_v \setminus A_v) \leq q \cdot \mathbf{H} + r$ for some $q \in \mathbf{N}^+$ and $r \in \mathbf{N}$ such that $q \geq v$. Accordingly, we get from axioms (F3)-(F4^b) and [11, Proposition 9] that

$$\mu^*(B_v) \leq \mu^*(A_v) + \mu^*(B_v \setminus A_v) \leq \mu^*(A_v) + \mu^*(q \cdot \mathbf{H} + r) = \mu^*(A_v) + \frac{1}{q} \leq \mu^*(A_v) + \frac{1}{v},$$

which in turn implies that $k > (\mu^*(B_v) - \mu^*(A_v))^{-1} \geq v$, i.e. $k \geq v + 1$. On the other hand, it follows from Lemma 1 that there exist $\mathcal{H}_0 \subseteq \llbracket 0, k - 1 \rrbracket$ and $h_0 \in \mathbf{N}$ such that

$$\mu^*(A_v \cup (B_v \cap \mathcal{V}_{k, \mathcal{H}_0})) < a \leq \mu^*(A_v \cup (B_v \cap \mathcal{V}_{k, \mathcal{H}_0 \cup \{h_0\}})). \quad (2)$$

Therefore, set $A_{v+1} := A_v \cup (B_v \cap \mathcal{V}_{k, \mathcal{H}_0})$ and $B_{v+1} := A_v \cup (B_v \cap \mathcal{V}_{k, \mathcal{H}_0 \cup \{h_0\}})$. Then, it is clear that $A_v \subseteq A_{v+1} \subseteq B_{v+1} \subseteq B_v$ and $B_{v+1} \setminus A_{v+1} \subseteq k \cdot \mathbf{H} + h_0$, which, putting it all together, is enough (by induction) to conclude, since $X \subseteq A_1$, $B_1 \subseteq Y$ and $k \geq v + 1$. \blacksquare

Now, let $(A_n)_{n \geq 1}$ and $(B_n)_{n \geq 1}$ be as in the above claim, and set $A := \bigcup_{n \geq 1} A_n$. For a fixed $n \in \mathbf{N}^+$, both $A \setminus A_n$ and $B_n \setminus A$ are then subsets of $B_n \setminus A_n$, which, by condition (iii), is in turn contained in $k \cdot \mathbf{H} + h$ for some $h \in \mathbf{N}$ and $k \geq n$.

Accordingly, we get from here, [11, Proposition 9] and axiom (F4^b) that $\mu^*(A \setminus A_n) \leq \frac{1}{n}$ and $\mu^*(B_n \setminus A) \leq \frac{1}{n}$ for all n , with the result that

$$\mu^*(A) \leq \mu^*(A_n) + \mu^*(A \setminus A_n) \leq \mu^*(A_n) + \frac{1}{n} < a + \frac{1}{n}$$

and

$$\mu^*(A) \geq \mu^*(B_n) - \mu^*(B_n \setminus A) \geq \mu^*(B_n) - \frac{1}{n} \geq a - \frac{1}{n},$$

where, along with the subadditivity of μ^* , we have used that $\mu^*(A_n) < a \leq \mu^*(B_n)$ by condition (ii). Hence, $|\mu^*(A) - a| \leq \frac{1}{n}$ for every $n \in \mathbf{N}^+$, which is possible if and only if $\mu(A) = a$, and completes the proof when considering that $X \subseteq A \subseteq Y$. \blacksquare

5. SHARPNESS OF MAIN THEOREM

Through this section, we want to show that the hypotheses of our main theorem are sharp, in the sense that if we try to extend the theorem to a class of functions $f : \mathcal{P}(\mathbf{H}) \rightarrow \mathbf{R}$ that is larger than the one of upper quasi-densities by dropping either of conditions (F3)-(F5) and replacing it with (F2), then the result breaks down in a somewhat dramatic way.

To this end, we will conveniently regard axioms (F1), ..., (F5) as words of a suitable formal language and let \mathcal{G} be a subset of $\mathcal{F} := \{(\text{F1}), \dots, (\text{F5})\}$; in particular, we write \mathcal{F}_1 for $\mathcal{F} \setminus \{(\text{F1})\}$, \mathcal{F}_2 for $\mathcal{F} \setminus \{(\text{F2})\}$, and so forth.

We denote by $\mathcal{N}^*(\mathcal{G})$ the set of all functions $f : \mathcal{P}(\mathbf{H}) \rightarrow \mathbf{R}$ that satisfy each of the axioms in \mathcal{G} along with the conditions $\text{Im}(\mu^*) \subseteq [0, 1]$ and $f(\emptyset) = 0$; e.g., $\mathcal{N}^*(\mathcal{F}_2)$ is the set of all upper quasi-densities (on \mathbf{H}), and $\mathcal{N}^*(\mathcal{F})$ the set of all upper densities.

Incidentally, we have from [11, Proposition 2] that if f is a function $\mathcal{P}(\mathbf{H}) \rightarrow \mathbf{R}$ that satisfies axioms (F1)-(F3), then its image is necessarily a subset of $[0, 1]$, while if f satisfies (F3), then $f(\emptyset) = 0$. However, things can be different in other cases, which hopefully explains the definition of $\mathcal{N}^*(\mathcal{G})$.

Building on these premises, we show by a series of three examples that if \mathcal{G} is either of \mathcal{F}_3 , \mathcal{F}_4 , or \mathcal{F}_5 , then there exists a function $f \in \mathcal{N}^*(\mathcal{G})$ such that the image of f is “as far as possible” from being the entire interval $[0, 1]$.

In this respect, we note that if f is a function $\mathcal{P}(\mathbf{H}) \rightarrow \mathbf{R}$ that satisfies axioms (F1)-(F3), then its image is a subset of $[0, 1]$. Moreover, if $f \in \mathcal{N}^*(\mathcal{G})$ and (F4) belongs to \mathcal{G} , then $f(k \cdot \mathbf{H}) = \frac{1}{k}$ for every $k \in \mathbf{N}^+$, hence $\{0\} \cup \{1/k : k \in \mathbf{N}^+\} \subseteq \text{Im}(f)$.

EXAMPLE 1. Let f be the function $\mathcal{P}(\mathbf{H}) \rightarrow \mathbf{R}$ defined as follows: Given $X \subseteq \mathbf{H}$, we assume $f(X) := 0$ if $|X^+| < \infty$, otherwise $f(X) := \sup_{n \geq 1} (x_{n+1} - x_n)^{-1}$, where $(x_n)_{n \geq 1}$ is the natural enumeration of X^+ . Of course, $f(\emptyset) = 0$, and it is straightforward that f satisfies conditions (F1), (F2) and (F4^b), and $\text{Im}(f) = \{0\} \cup \{1/k : k \in \mathbf{N}^+\}$.

EXAMPLE 2. Let f be a monotone and subadditive function $\mathcal{P}(\mathbf{H}) \rightarrow \mathbf{R}$ such that $f(\mathbf{H}) = 1$ and $f(\emptyset) = 0$ (e.g., f may be an upper density on \mathbf{H}), and let f be the function $\mathcal{P}(\mathbf{H}) \rightarrow \mathbf{R}$ mapping a set $X \subseteq \mathbf{H}$ to 1 if $f(X) > 0$, and to 0 otherwise, cf. [11, Example 1].

It is seen that f satisfies axioms (F1)-(F3) and (F5), where we use, in particular, that $f(X \cup Y) \leq f(X) + f(Y)$ for all $X, Y \subseteq \mathbf{H}$, and hence $f(X \cup Y) = 0$ whenever $f(X) = f(Y) = 0$. On the other hand, it is evident that $f(\emptyset) = 0$ and the image of f is the 2-element set $\{0, 1\}$.

EXAMPLE 3. Let f be the function $f : \mathcal{P}(\mathbf{H}) \rightarrow \mathbf{R} : X \mapsto (\inf(X^+))^{-1}$, where $\inf(\emptyset) := \infty$ and $\frac{1}{\infty} := 0$. It is immediate that f satisfies conditions (F1)-(F4), cf. [11, Example 3]. In addition, $f(\emptyset) = 0$ and the image of f is the set $\{0\} \cup \{1/k : k \in \mathbf{N}^+\}$.

Next, we show that if \mathcal{G} is either of \mathcal{F}_3 , \mathcal{F}_4 , or \mathcal{F}_5 , then we can find a function $f \in \mathcal{N}^*(\mathcal{G})$ whose image is the entire interval $[0, 1]$. This is based on a second series of three examples (which are sensibly more difficult to construct than the previous ones), and gives further evidence of the sharpness of the assumptions made in the statement of the main theorem.

EXAMPLE 4. Let f be the same as in Example 1, and let g be the function $\mathcal{P}(\mathbf{H}) \rightarrow \mathbf{R}$ defined as follows: Given $X \subseteq \mathbf{H}$, we assume $g(X) := 0$ if $|X^+| < \infty$, otherwise

$$g(X) := \liminf_{n \rightarrow \infty} \frac{x_n}{x_{n+1}},$$

with $(x_n)_{n \geq 1}$ being the natural enumeration of X^+ . It is seen that $g(\mathbf{H}) = 1$ and $g(k \cdot X + h) = g(X)$ for all $X \subseteq \mathbf{H}$, $k \in \mathbf{N}^+$ and $h \in \mathbf{N}$, and it is not difficult to check that g is monotone, by [1, Theorem 18.3(e)] and the fact that $\frac{x}{z} \leq \frac{y}{z}$ if $x, y, z \in \mathbf{R}$ and $0 < x \leq y \leq z$.

Let ϕ denote the function $\mathcal{P}(\mathbf{H}) \rightarrow \mathbf{R} : X \mapsto f(X)g(X)$. It follows from the above and Example 1 that $\phi(\emptyset) = 0$ and ϕ satisfies axioms (F1), (F2) and (F4^b).

Now, fix $a \in [1, \infty[$ and define an integral sequence $(x_n)_{n \geq 1}$ by taking $x_1 := 1$, $x_2 := 2$ and $x_{n+1} := \lceil ax_n \rceil$ for $n \geq 2$; then set $X_a := \{x_n : n \in \mathbf{N}^+\}$. It is easy to check that $f(X_a) = 1$ and $g(X_a) = \frac{1}{a}$, and hence $\phi(X_a) = \frac{1}{a}$. This yields that $\text{Im}(\phi) = [0, 1]$, because $\phi(\emptyset) = 0$ and every real number in the interval $]0, 1[$ is the reciprocal of some element in $[1, \infty[$.

On the other hand, we are going to prove that ϕ does not have the weak Darboux property. In fact, consider the set $X := \{2^n : n \in \mathbf{N}\}$, so that $f(X) = 1$ and $\phi(X) = g(X) = \frac{1}{2}$; we want to show that $\phi(Y)$ does not attain the value $\frac{3}{8}$ for $Y \subseteq X$.

To this end, let $Y \subseteq X$. If Y is finite, then $\phi(Y) = 0$, and we are done. So assume $|Y| = \infty$, and let $(x_n)_{n \geq 1}$ and $(y_n)_{n \geq 1}$ be, respectively, the natural enumerations of X and Y .

If $X \setminus Y$ is infinite, then $y_{n+1} \leq 4y_n$ for infinitely many n , and hence $\phi(Y) \leq g(Y) \leq \frac{1}{4} < \frac{3}{8}$. Otherwise, $g(Y) = g(X)$ and $\phi(Y) = \frac{1}{2}f(Y)$, which in turn implies that either $\phi(Y) = \frac{1}{2}$ or $\phi(Y) \leq \frac{1}{4}$, since $\text{Im}(f) \subseteq \{0\} \cup \{1/k : k \in \mathbf{N}^+\}$. In any case, $\phi(Y) \neq \frac{3}{8}$.

EXAMPLE 5. Let \mathbf{d}^* and \mathbf{b}^* be, respectively, the upper asymptotic and upper Buck densities on \mathbf{H} , and consider the function

$$f : \mathcal{P}(\mathbf{H}) \rightarrow \mathbf{R} : X \mapsto \begin{cases} 1 & \text{if } \mathbf{d}^*(X) > 0 \\ \mathbf{b}^*(X) & \text{otherwise} \end{cases}.$$

Since $\mathbf{d}^*(\mathbf{H}) = 1$, and both of \mathbf{d}^* and \mathbf{b}^* are upper densities (thus, in particular, monotone and shift-invariant), it is seen that f satisfies (F1), (F2) and (F5); also, $f(\emptyset) = 0$ and $\text{Im}(f) \subseteq [0, 1]$.

We want to show that f is subadditive. For this, pick $X, Y \subseteq \mathbf{H}$. If $\mathbf{d}^*(X) > 0$ or $\mathbf{d}^*(Y) > 0$, then $f(X) + f(Y) \geq 1 \geq f(X \cup Y)$, and we are done. Otherwise, it follows from the subadditivity of \mathbf{d}^* that $\mathbf{d}^*(X \cup Y) = \mathbf{d}^*(X) = \mathbf{d}^*(Y) = 0$, and then $f(X \cup Y) = \mathbf{b}^*(X \cup Y) \leq \mathbf{b}^*(X) + \mathbf{b}^*(Y) = f(X) + f(Y)$ by the subadditivity of \mathbf{b}^* . Therefore, f satisfies (F3), as was to prove the case.

Now, let $X := \{n + n! : n \in \mathbf{N}\} \subseteq \mathbf{N}^+$. It is immediate that $d^*(X) = 0$, hence $f(Y) = b^*(Y)$ for every $Y \subseteq X$ (by the monotonicity of d^*). On the other hand, we get from Theorem 3 that b^* has the strong Darboux property, so f takes any value between 0 and $b^*(X)$, and since it is known, see [16, Theorem 3.2], that $b^*(X) = 1$, we obtain that $\text{Im}(f) = [0, 1]$.

However, f does not have the weak Darboux property: In fact, $f(2 \cdot \mathbf{H}) = 1$, but if $X \subseteq 2 \cdot \mathbf{H}$ then either $d^*(X) > 0$, and hence $f(X) = 1$, or $f(X) = b^*(X) \leq b^*(2 \cdot \mathbf{H}) = \frac{1}{2}$, where we use that b^* is monotone and (-1) -homogeneous. To wit, $f(Y)$ does not attain any value strictly in between $\frac{1}{2}$ and 1 as Y ranges over the subsets of $2 \cdot \mathbf{H}$, which is enough to conclude.

EXAMPLE 6. Let \mathfrak{s}^* be the function $\mathcal{P}(\mathbf{H}) \rightarrow \mathbf{R} : X \mapsto \sup_{n \geq 1} \frac{1}{n} |X \cap [1, n]|$, which we refer to as the upper Schnirelmann density on \mathbf{H} , cf. [8, p. 18].

By general considerations made in Section 3, we see that \mathfrak{s}^* does not have the weak Darboux property, because $\mathfrak{s}^*(X) > 0$ for *every* nonempty set $X \subseteq \mathbf{N}^+$. However, we are going to show that \mathfrak{s}^* satisfies axioms (F1)-(F4) and $\text{Im}(\mathfrak{s}^*)$ is the entire interval $[0, 1]$.

It is straightforward that $\mathfrak{s}^*(\mathbf{H}) = 1$ and \mathfrak{s}^* is monotone and subadditive. As for (F4), let $k \in \mathbf{N}^+$ and $X \subseteq \mathbf{H}$. For every $\varepsilon \in \mathbf{R}^+$ there exists $n_\varepsilon \in \mathbf{N}^+$ such that

$$\mathfrak{s}^*(X) - k\varepsilon < \frac{|X \cap [1, n_\varepsilon]|}{n_\varepsilon} \leq \mathfrak{s}^*(X).$$

which, taking into account that $|X \cap [1, n_\varepsilon]| = |(k \cdot X) \cap [1, kn_\varepsilon]|$, yields $\mathfrak{s}^*(k \cdot X) \geq \frac{1}{k} \mathfrak{s}^*(X)$. Now, to prove the opposite inequality and thus obtain that \mathfrak{s}^* is (-1) -homogeneous, fix $n \in \mathbf{N}^+$ and write $n = qk + r$, where $q \in \mathbf{N}$ and $r \in \llbracket 0, n - 1 \rrbracket$. Then we have

$$\frac{|(k \cdot X) \cap [1, n]|}{n} = \frac{|(k \cdot X) \cap [1, qk]|}{n} = \frac{|X \cap [1, q]|}{qk + r} \leq \frac{|X \cap [1, q]|}{qk} \leq \frac{1}{k} \mathfrak{s}^*(X),$$

which implies $\mathfrak{s}^*(k \cdot X) \leq \frac{1}{k} \mathfrak{s}^*(X)$, and hence leads to the desired conclusion.

We are left to show that $\text{Im}(\mathfrak{s}^*) = [0, 1]$. To this end, pick a real $a \geq 1$ and let X_a be the set of all positive integers of the form $\lceil na \rceil$ as n ranges over \mathbf{N}^+ .

Let d^* be the upper asymptotic density (on \mathbf{H}). Since $\lceil na \rceil < \lceil (n+1)a \rceil$ for all n , it follows from well-known formulas, see, e.g., [15, Section 2], that

$$d^*(X_a) = \limsup_{n \rightarrow \infty} \frac{n}{\lceil na \rceil} = \frac{1}{a}.$$

Therefore, we have $\mathfrak{s}^*(X_a) \geq \frac{1}{a}$, as it is clear from the basic properties of the upper limit of a real sequence, see, e.g., [1, Theorem 18.3(e)], that $d^*(X) \leq \mathfrak{s}^*(X)$ for every $X \subseteq \mathbf{H}$.

On the other hand, however we choose $n \in \mathbf{N}^+$, it holds that either $|X_a \cap [1, n]| = 0$ if $n < \lceil a \rceil$, or there is a maximum $q \in \mathbf{N}^+$ such that $\lceil qa \rceil \leq n$ otherwise, in which case it is found that

$$\frac{|X \cap [1, n]|}{n} = \frac{q}{n} \leq \frac{q}{\lceil qa \rceil} \leq \frac{1}{a}.$$

It follows that $\mathfrak{s}^*(X_a) = \frac{1}{a}$. But every real number in the interval $]0, 1]$ is the reciprocal of some $a \in [1, \infty[$ and, of course, $\mathfrak{s}^*(\emptyset) = 0$, so we can conclude from the above that $\text{Im}(\mathfrak{s}^*) = [0, 1]$.

Additional results in the same spirit will be considered in a subsequent paper, as they involve fairly intricate constructions, which would lead us too far from the scope of the present work.

6. CLOSING REMARKS

Suppose that μ^* is a function $\mathcal{P}(\mathbf{H}) \rightarrow \mathbf{R}$, and denote by μ the partial function $\mathcal{P}(\mathbf{H}) \rightarrow \mathbf{R}$ obtained by restriction of μ^* to the set

$$\{X \subseteq \mathbf{H} : \mu^*(X) + \mu^*(X^c) = 1\} = \{X \subseteq \mathbf{H} : \mu^*(X) = \mu_*(X)\} \subseteq \mathcal{P}(\mathbf{H}).$$

If, in particular, μ^* is an upper [quasi-]density, then μ is called the [quasi-]density induced by μ^* , or just a [quasi-]density (on \mathbf{H}) if explicit reference to μ^* is unnecessary.

We get by [11, Proposition 2(vi)] that $X \in \text{dom}(\mu)$ whenever μ^* is monotonic and subadditive and $X \subseteq \mathbf{H}$ is such that $\mu^*(X) = 0$ or $\mu_*(X) = 1$. Moreover, $\text{dom}(\mu)$ is closed under complementation, and we have from [11, Proposition 7] that $X \in \text{dom}(\mu)$ if μ is an upper density and X differs from a finite union of arithmetic progressions of \mathbf{H} by finitely many integers.

Now assume for the remainder that μ^* is an upper quasi-density. In the light of Theorem 3, we find it natural to ask whether μ must have the strong Darboux property: It occurs that the answer is affirmative if μ^* is additive, as in that case $\text{dom}(\mu)$ is nothing but $\mathcal{P}(\mathbf{H})$, and hence Theorem 3 gives that μ has the strong Darboux property. (Additive upper quasi-densities are monotone, and from [11, Remark 3] the existence of additive upper densities is provable in ZFC, but independent of ZF.) However, we ignore what happens in general, so we raise the following question, which has a positive answer in the case when μ^* is either the upper Buck density or the upper Banach density, see [16, Theorem 2.1] and [5, Theorem 4.2], respectively.

Question 1. With the same notation as above, does μ need to have the strong Darboux property? If not, what about the weak Darboux property? And does the answer change if we assume μ^* to be monotone (viz., an upper density)?

A few more questions along these lines can be found in [11], see in particular [11, Question 4], and involve a kind of “joint weak Darboux property” in the spirit of [9], where the focus is, however, on (upper and lower) weighted densities.

With this said, let a function $f : \mathcal{P}(\mathbf{H}) \rightarrow \mathbf{R}$ have the *symmetric (strong) Darboux property* if for all $X, Y \subseteq \mathbf{H}$ with $X \subseteq Y$ and every $a \in [0, 1]$ there exists a set A such that $X \subseteq A \subseteq Y$ and $f(A) = af(Y) + (1 - a)f(X)$; this is the same as the strong Darboux property when f is monotone, but is more general otherwise, as it does not require any longer that $f(X) \leq f(Y)$.

So, it seems natural to ask if Theorem 3 can be extended to prove that upper quasi-densities satisfy the symmetric Darboux property. Yet, that is not the case, as we are going to show: To start with, let θ^* be the function

$$\theta^* : \mathcal{P}(\mathbf{H}) \rightarrow \mathbf{R} : X \mapsto \begin{cases} d^*(X) & \text{if } \iota(X) \leq 3 \\ \frac{3}{4}(\iota(X) - 1)d^*(X) & \text{if } 4 \leq \iota(X) < \infty, \\ 0 & \text{otherwise} \end{cases},$$

where d^* is the upper asymptotic density on \mathbf{Z} and ι the function $\mathcal{P}(\mathbf{H}) \rightarrow \mathbf{N}^+ \cup \{\infty\}$ taking a set $X \subseteq \mathbf{H}$ to the infimum of the integers $n \geq 1$ for which there is $Y \subseteq \mathbf{H}$ such that $d^*(Y) \geq \frac{1}{n}$ and $|(q \cdot Y + r) \setminus X| < \infty$ for some $q \in \mathbf{N}^+$ and $r \in \mathbf{Z}$, with the convention that $\inf(\emptyset) := \infty$.

It was already observed in [11], in reference to [11, Question 6], that θ^* is a non-monotone upper quasi-density, so here we just prove that it does not have the symmetric Darboux property.

For this, set $X := \bigcup_{n \geq 2} \llbracket \frac{1}{4}(2n-1)! + \frac{3}{4}(2n)! + 1, (2n)! + 1 \rrbracket$ and $Y := X \cup (4 \cdot \mathbf{H})$; note that we are intentionally starting with $n = 2$ in the definition of X , so as to discard the integers ≤ 3 and have that Y can be written as the union of $4 \cdot \mathbf{H}$ and $\bigcup_{h=1}^3 (X \cap (4 \cdot \mathbf{H} + h))$, which otherwise would be false for $\mathbf{H} = \mathbf{N}^+$.

Then, using that d^* satisfies (F3) and (F4^b) yields, together with [11, Lemma 1], that $d^*(X) = \frac{1}{4}$ and $d^*(Y) \leq d^*(4 \cdot \mathbf{H}) + \sum_{h=1}^3 d^*(X \cap (4 \cdot \mathbf{H} + h)) = \frac{1}{4} + \frac{3}{16} = \frac{7}{16}$.

On the other hand, it follows from the claim established within the proof of [11, Lemma 3] and the observation made on the first line of the same that $\iota(X) = 4$ and $\iota(Y) = 1$.

Now, consider a set A such that $X \subseteq A \subseteq Y$. By [11, Lemma 3], we have $\iota(A) \leq \iota(X) = 4$, and so there are two cases: Either $\iota(A) \leq 3$, and then $\theta^*(A) = d^*(A) \leq d^*(Y) \leq \frac{7}{16} < \frac{1}{2}$, or $\iota(A) = 4$, and then $\theta^*(A) = \frac{9}{4}d^*(A) \geq \frac{9}{4}d^*(X) = \frac{9}{16} > \frac{1}{2}$. This shows that $\theta^*(A)$ cannot attain the value $\frac{1}{2}$, so θ^* does not have the symmetric Darboux property.

ACKNOWLEDGMENTS

P.L. is supported by a PhD scholarship from Università “Luigi Bocconi”, and S.T. by NPRP grant No. [5-101-1-025] from the Qatar National Research Fund (a member of Qatar Foundation). The statements made herein are solely the responsibility of the authors.

The authors are grateful to Ladislav Mišík (University of Ostrava, SK) for having suggested Example 1, to Carlo Sanna (Università di Torino, IT) for many useful comments, and to Nordine Mir (Texas A&M University at Qatar, Q) for fruitful discussions.

REFERENCES

- [1] R. G. Bartle, *The Elements of Real Analysis*, John Wiley & Sons, 1976 (2nd edition).
- [2] K. P. S. Bhaskara Rao and M. Bhaskara Rao, *Theory of Charges: A Study of Finitely Additive Measures*, Pure and Applied Mathematics **109**, London: Academic Press, 1983.
- [3] N. Dinculeanu, *Vector Measures*, International Series of Monographs in Pure and Applied Mathematics **95**, Oxford: Pergamon Press, 1966.
- [4] G. Fichtenholz, *Sur les fonctions d'ensemble additives et continues*, Fund. Math. **7** (1925), No. 1, 296–301 (in French).
- [5] Z. Gáliková, B. László, and T. Šalát, *Remarks on uniform density of sets of integers*, Acta Acad. Paed. Agriensis, Sectio Mathematicae **29** (2002), 3–13.
- [6] G. Grekos, *Sur la répartition des densités des sous-suites d'une suite d'entiers*, PhD thesis, Université Pierre et Marie Curie, 1976.
- [7] ———, *Répartition des densités des sous-suites d'une suite d'entiers*, J. Number Theory **10** (1978), No. 2, 177–191.
- [8] ———, *On various definitions of density (survey)*, Tatra Mt. Math. Publ. **31** (2005), 17–27.
- [9] G. Grekos, L. Mišík, and J. T. Tóth, *Density sets of sets of positive integers*, J. Number Theory **130** (2010), No. 6, 1399–1407.
- [10] K. Kuratowski, *Topology: Volume I*, London: Academic Press, 1966.
- [11] P. Leonetti and S. Tringali, *On the notions of upper and lower density*, preprint, last updated: Oct 05, 2015 ([arXiv:1506.04664](https://arxiv.org/abs/1506.04664)).

-
- [12] M. Mačaj, L. Mišík, and J. Tomanová, *On a class of densities of sets of positive integers*, Acta Math. Univ. Comenianae **72** (2003), No. 2, 213–221.
 - [13] D. Maharam, *The representation of abstract integrals*, Trans. Amer. Math. Soc **75** (1953), 154–184.
 - [14] ———, *Finitely additive measures on the integers*, Sankhyā Ser. A **38** (1976), No. 1, 44–59.
 - [15] I. Niven, *The asymptotic density of sequences*, Bull. Amer. Math. Soc. **57** (1951), No. 6, 420–434.
 - [16] M. Paštéka and T. Šalát, *Buck’s measure density and sets of positive integers containing arithmetic progression*, Math. Slovaca **41** (1991), No. 3, 283–293.
 - [17] W. Sierpiński, *Sur les fonctions d’ensemble additives et continues*, Fund. Math. **3** (1922), No. 1, 240–246 (in French).
 - [18] R. Sikorski, *Funkcje rzeczywiste, Tom I*, PWN: Warsaw, 1958 (in Polish).
 - [19] A. Soifer, *The Mathematical Coloring Book*, Springer-Verlag: New York, 2009.

UNIVERSITÀ “LUIGI BOCCONI”, VIA SARFATTI 25, 20136 MILANO, ITALY
E-mail address: leonettipaolo@gmail.com

DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY AT QATAR, PO BOX 23874 DOHA, QATAR
E-mail address: salvo.tringali@gmail.com
URL: <http://www.math.polytechnique.fr/~tringali/>